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ON THE PERIODIC RESONANT SOLUTIONS OF THE HAMILTONIAN SYSTEMS GENERATING FROM THE POSITION OF EQUILIBRIUM*

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A method is proposed of constructing and investigating the stability of periodic solutions of a canonical system of differential equations generating from the position of equilibrium. It is assumed that the system is almost autonomous and that a resonance exists in the forced oscillations. The investigation is based on applying to the Hamiltonian function a series of canonical changes of variables containing a small parameter. These changes make it possible to pass, in the finite approximation with respect to the small parameter, from the initial nonautonomous system with n degrees of freedom to an autonomous system with one degree of freedom. A constructive procedure is given for formation and study of the periodic solution, requiring only the computation of the coefficients of the normal form of the Hamiltonian function.

1. Let us consider a system of canonical differential equations

$$\frac{d\xi_i}{dt}=\frac{\partial\Gamma}{\partial\eta_i},\quad \frac{d\eta_i}{dt}=-\frac{\partial\Gamma}{\partial\xi_i}\quad (i=1,\ldots,n) \tag{1.1}$$

Here ξ_i, η_i are the canonically conjugated coordinates and impulses, and t is an independent variable. We shall assume that the Hamiltonian function Γ depends on the small parameter μ. When $\mu=0$, it determines the autonomous system of differential equations with the position of equilibrium $\xi_i=\eta_i=0$. Let the function Γ be also analytic relatively to ξ_i, η_i and μ in the neighborhood of their zero values, where it can be written in the form

$$\Gamma=\Gamma_2+\Gamma_3+\Gamma_4+\mu\Gamma_1+\ldots \tag{1.2}$$

$$\Gamma_p=\sum_{\nu_1+\ldots+\nu_n+\mu_1+\ldots+\mu_n=p}\gamma_{\nu_1\ldots\nu_n\mu_1\ldots\mu_n}\xi_1^{\nu_1}\ldots\xi_n^{\nu_n}\eta_1^{\mu_1}\ldots\eta_n^{\mu_n}\quad (p=2,3,4,\ldots)$$

$$\Gamma_1=\Gamma^{(1)}\sum_{k=1}^{\infty}A_k\sin kt+\Gamma^{(2)}\sum_{k=1}^{\infty}B_k\cos kt$$

$$\Gamma^{(s)}=\sum_{i=1}^{n}(\alpha_{si}\xi_i+\beta_{si}\eta_i)\quad (s=1,2)$$

In the Hamiltonian (1.2) the dots denote terms of at least third order in μ, ξ_i, η_i and at least fifth order in ξ_i, η_i ; the coefficients $\gamma_{\nu_1\ldots\nu_n\mu_1\ldots\mu_n}$, A_k, B_k, α_{si}, β_{si} are constants and ν_i, μ_i are nonnegative integers. We assume that when $\mu=0$, then the system (1.1) contains no resonances up to and including the fourth order, i.e. that the following relation holds:

$$\sum_{i=1}^{n}m_i\omega_i\neq 0\quad \Big(0<\sum_{i=1}^{n}|m_i|\leqslant 4\Big) \tag{1.3}$$

for all integral m_i satisfying the inequality within the brackets. Here ω_i denote the eigenfrequencies of the system linearized in the neighborhood of the position of equilibrium and defined by the Hamiltonian Γ_2. Then a real canonical transformation $\xi_i,\eta_i\to q_i,p_i,\Gamma\to H$ exists analytic in ξ_i, η_i and reducing the Hamiltonian function (1.2) to its normal form /1,2/:

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$$H = H_2 + H_4 + \mu H_1 + \dots \tag{1.4}$$
$$H_2 = \frac{1}{2}\sum_{i=1}^{n} \sigma_i\omega_i (q_i^2 + p_i^2), \quad H_4 = \frac{1}{4}\sum_{i,j=1}^{n} l_{ij}(q_i^2 + p_i^2)(q_j^2 + p_j^2)$$
$$H_1 = H^{(1)} \sum_{k=1}^{\infty} A_k \sin kt + H^{(2)} \sum_{k=1}^{\infty} B_k \cos kt$$
$$H^{(s)} = \sum_{i=1}^{n} (a_{si}q_i + b_{si}p_i) \quad (s = 1, 2)$$

Here l_{ij}, a_{si}, b_{si} are constants coefficients and $l_{ij} = l_{ji}$, and the quantities $\sigma_i = \pm 1$ are obtained in the course of normalization of Γ_2.

Let us first consider the system described by the Hamiltonian function $H_2 + \mu H_1$ $(\mu \neq 0)$. Then the corresponding equations are linear and admit the following particular periodic solution:

$$q_i^* = \mu \sum_{k=1}^{\infty} \frac{1}{k^2 - \omega_i^2} [(\sigma_i\omega_i a_{1i} A_k + k b_{2i} B_k) \sin kt + (\sigma_i\omega_i a_{2i} B_k - k b_{1i} A_k) \cos kt] \tag{1.5}$$
$$p_i^* = \mu \sum_{k=1}^{\infty} \frac{1}{k^2 - \omega_i^2} [(\sigma_i\omega_i b_{1i} A_k - k a_{2i} B_k) \sin kt + (\sigma_i\omega_i b_{2i} B_k + k a_{1i} A_k) \cos kt]$$

From this it follows that if the frequency $\omega_m = N_0$, where N_0 is a natural number, and at least one of the quantities

$$f(\sigma_m) = -\sigma_m A_{N_0} b_{1m} + B_{N_0} a_{2m}, \quad g(\sigma_m) = \sigma_m A_{N_0} a_{1m} + B_{N_0} b_{2m} \tag{1.6}$$

does not vanish, then the formulas (1.5) do not describe the motion in a linear system. In this case we have a resonance in the forced oscillations.

The problem of constructing periodic solutions in the case of a resonance appearing in the forced oscillations has been studied in great detail in /3 – 11/ for the general type of differential equations systems. Below we propose a method of analysing the periodic solutions of the Hamiltonian systems which are almost autonomous. An algorithm is given for construction and study of periodic solutions, based on the normalization of the Hamiltonian function. The analysis of the solution requires only the knowledge of the coefficients of the normal form. Moreover the problem of normalizing the Hamiltonian function does not present any difficulties. The classical Birkhoff algorithm /1/ is well known. At present the DePree – Hori normalization algorithm is widely used. A number of programs have been written for this algorithm, for computing the coefficients of the normal form on a digital computer (*). We note that the problem of investigating the periodic solutions of the almost autonomous Hamiltonian systems was studied earlier (**) under the assumption that the Hamiltonian functions have symmetry properties.

2. Let $\omega_m = N_0 + \varepsilon$ $(|\varepsilon| \ll 1)$ and ω_i $(i = 1, \dots, n;\ i \neq m)$ have no integers, and let at least one of the quantities (1.6) be nonzero. To obtain a periodic solution in this resonant case, we must take into account the terms of the equation of motion nonlinear in q_i and p_i. We perform, in the system with the Hamiltonian (1.4), the variable change $q_i = q_i^* + q_i'$, $p_i = p_i^* + p_i'$ $(i = 1, \dots, n)$ where q_i^* and p_i^* are given by the formula (1.5) and the terms containing $k = N_0$ are eliminated from the sums (1.5) where $i = m$. The above variables change cancels in the Hamiltonian H_1 the nonresonant terms (of frequency different from N_0). In the variables q_i', p_i' the Hamiltonian function assumes the following form:

$$H = \frac{1}{2}\sum_{i=1}^{n} \sigma_i\omega_i (q_i'^2 + p_i'^2) + \frac{1}{4}\sum_{i,j=1}^{n} l_{ij}(q_i'^2 + p_i'^2)(q_i'^2 + p_j'^2) + \mu [A_{N_0}(a_{1m}q_m' + b_{1m}p_m') \sin N_0 t + B_{N_0}(a_{2m}q_m' + b_{2m}p_m') \cos N_0 t] + \dots \tag{2.1}$$

*) Markeev A.P. and Sokol'skii A.G. Certain computational algorithms for normalizing the Hamiltonian systems. Preprint In-ta prikl. matematiki Akad. Nauk SSSR, Moscow, No.31, 1976.

**) Sazonov V.V. and Sarychev V.A. Periodic solutions of the almost autonomous systems of ordinary differential equations. Preprint In-ta prikl. matematiki Akad. Nauk SSSR, Moscow, No. 90, 1977.

In the approximate system with the Hamiltonian (2.1) in which we neglect all terms which have not been written out, we have the solution $q_i'=p_i'=0$ $(i=1,\ldots,n;\ i\neq m)$, and the variation in q_m' and p_m' is determined by the Hamiltonian function

$$H'={}^1/_2\,\sigma_m\omega_m(q_m'^2+p_m'^2)+{}^1/_4\,l_{mm}(q_m'^2+p_m'^2)^2+ \\ \mu\,[A_{N_0}(a_{1m}q_m'+b_{1m}p_m')\sin N_0t+B_{N_0}(a_{2m}q_m'+b_{2m}p_m')\cos N_0t] \tag{2.2}$$

Passing to polar coordinates in accordance with the formulas $q_m'=\sqrt{2r}\sin\varphi,\ p_m'=\sqrt{2r}\cos\varphi$, we obtain (2.2) in the form

$$H=\sigma_m\omega_m r+l_{mm}r^2+{}^1/_2\mu\sqrt{2r}\,[f(-1)\sin(\varphi+N_0t)+ \\ g(-1)\cos(\varphi+N_0t)+f(+1)\sin(\varphi-N_0t)+g(+1)\times \\ \cos(\varphi-N_0\,t)] \tag{2.3}$$

where the quantities $f(\sigma_m),\ g(\sigma_m)$ $(\sigma_m=\pm1)$ are obtained from the formulas (1.6).

The variable change $r,\ \varphi\to\rho,\ \theta$ defined by the generating function

$$S=\rho\varphi-\mu\frac{\sigma_m\sqrt{2\rho}}{4N_0}\,[g(-\sigma_m)\sin(\varphi+\sigma_mN_0t)-f(-\sigma_m)\cos(\varphi+\sigma_mN_0t)]$$

according to the formulas

$$r=\frac{\partial S}{\partial\varphi},\qquad \theta=\frac{\partial S}{\partial\rho}$$

eliminates from the Hamiltonian (2.3) the terms containing $\sin(\varphi+\sigma_mN_0t),\ \cos(\varphi+\sigma_mN_0t)$. In the variables ρ,θ the approximate system has the following Hamiltonian:

$$H=\sigma_m\omega_m\rho+l_{mm}\rho^2+{}^1/_2\mu\sqrt{2\rho}\ \ \delta\cos(\theta-\sigma_mN_0t+\theta_0) \\ \delta=\sqrt{f^2(\sigma_m)+g^2(\sigma_m)},\quad \sin\theta_0=-\frac{f(\sigma_m)}{\delta},\quad \cos\theta_0=\frac{g(\sigma_m)}{\delta} \tag{2.4}$$

Further, the substitution $\rho=R,\ \theta=\psi+\sigma_mN_0t-\theta_0$ yields the autonomous system

$$\frac{dR}{dt}=\mu\delta\frac{\sqrt{2R}}{2}\sin\psi,\qquad \frac{d\psi}{dt}=\sigma_m\varepsilon+2l_{mm}R+\mu\frac{\delta}{2\sqrt{2R}}\cos\psi \tag{2.5}$$

with Hamiltonian function

$$H=\sigma_m\varepsilon R+l_{mm}R^2+{}^1/_2\mu\sqrt{2R}\,\delta\cos\psi \tag{2.6}$$

If the solution R_0,ψ_0 of the system (2.5) corresponds to the position of equilibrium, then we obtain the following 2π-periodic solutions in the variables q_i and p_i :

$$q_m^\circ=\sqrt{2R_0}\sin(\sigma_mN_0t+\psi_0-\theta_0)+\ldots,\quad q_i^\circ=0+\ldots \\ (i=1,\ldots,n;\ i\neq m) \\ p_m^\circ=\sqrt{2R_0}\cos(\sigma_mN_0t+\psi_0-\theta_0)+\ldots,\quad p_i^\circ=0+\ldots \tag{2.7}$$

From (2.5) it follows that $\psi_0=0$ or $\psi_0=\pi$, and R_0 satisfies the equation

$$z^3+\frac{\sigma_m\varepsilon}{l_{mm}}z+\frac{\mu\delta}{2l_{mm}}=0\qquad (z\cos\psi_0=\sqrt{2R_0}) \tag{2.8}$$

where we assume that the coefficient $l_{mm}\neq0$.

In what follows we shall limit ourselves to investigating the periodic solutions passing, at $\mu=0$, to the position of equilibrium $\xi_i=\eta_i=0$ of the generating system. Equation (2.8) has either one, or three real roots, depending on whether its discriminant

$$D=\frac{1}{l_{mm}^2}\left(\frac{\mu^2\delta^2}{16}+\frac{\sigma_m\varepsilon^3}{27l_{mm}}\right)$$

is positive or negative, respectively. From (2.7) and (2.8) it follows that when $D<0$ $(D>0)$, then three (one) 2π-periodic solutions (solution) exist(s). The equation of the branching curve $D=0$ has the form

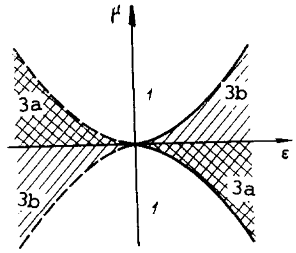

Fig.1

$$|\mu| = \frac{4}{3\delta}\sqrt{-\frac{\sigma_m \varepsilon^3}{3l_{mm}}}, \quad (\sigma_m \varepsilon l_{mm} < 0) \qquad (2.9)$$

The Fig.1 depicts schematically the curves $D = 0$ in the parameter plane ε, μ ($\sigma_m = +1$). Solid lines depict the curves at $l_{mm} < 0$, and the dashed lines at $l_{mm} > 0$. The shading shows the domains of existence of three periodic solutions.

Taking into account the notation introduced above, we can finally write the 2π-periodic solutions (2.7) in the form

$$q_m^\circ = z\sin(\sigma_m N_0 t - \theta_0) + \ldots, \quad q_i^\circ = 0 + \ldots \qquad (2.10)$$
$$(i = 1, \ldots, n; \quad i \neq m)$$
$$p_m^\circ = z\cos(\sigma_m N_0 t - \theta_0) + \ldots, \quad p_i^\circ = 0 + \ldots$$

where θ_0 and z are given by the formulas (2.4) and equations (2.8) respectively.

The expression (2.10) contains, in the explicit form, only the principal (of the order of $\mu^{1/3}$) terms, and repeated dots denote the terms of the order of at least $\mu^{2/3}$. To include in (2.10) terms of the order of $\mu^{2/3}$ and higher, we must restore in the Hamiltonian function (1.4) the neglected terms of the order of $\mu^{4/3}$ and higher respectively. If the coefficient l_{mm} in the Hamiltonian function H_4 of (1.4) is zero, then a sixth order form in ξ_i, η_i must be included in the Hamiltonian (1.2). In terms of the normal coordinates q_i and p_i this form is

$$H_6 = \frac{1}{8}\sum_{i,j,k=1}^{n} l_{ijk}(q_i^2 + p_i^2)(q_j^2 + p_j^2)(q_k^2 + p_k^2)$$

In addition, we must demand that, when $\mu = 0$, then the system (1.1) be free of resonances of up to and including the sixth order. Carrying out the computations analogous to the case $l_{mm} \neq 0$, we obtain the following equation for z from (2.10):

$$z^5 + \frac{4\sigma_m\varepsilon}{3l_{mmm}}z + \frac{2\mu\delta}{3l_{mmm}} = 0$$

3. A strict proof of the formal procedure of constructing the periodic solutions given in Sect.2 can be carried out with the help of the Poincaré's small parameter method /3 – 5/. It was found that the 2π-periodic solutions (2.10) of the system (1.1) constructed formally with the help of the normal forms, in the presence of resonance in the forced oscillations and for sufficiently small in module values of μ, indeed exist for all values of the parameters of the problem included in the discussion, except perhaps the branching curve $D=0$ given by formula (2.9). No other 2π-periodic solution passing at $\mu = 0$ to the solution $\xi_i = \eta_i = 0$ exists in the system (1.1).

We note that in the paper /4/, while studying periodic solutions in the systems close to the Liapunov systems, it was assumed that the quantity $\varepsilon = \omega_m - N_0$ was of the order of μ when a resonance was present, and the existence of a unique periodic solution passing at $\mu = 0$ to the position of equilibrium of the generating system, was established. In the present case we have either one, or three periodic solutions. The disparity in the number of periodic solutions is caused by the fact that here the parameters μ and ε are assumed independent. Three periodic solutions exist in the case when ε is of the order of $\mu^{2/3}$. If ε is of the first order with respect to μ, then the system (1.1) has, at sufficiently small in module values of μ only one periodic solution obtained in /4/.

4. Let us consider the problem of stability of the periodic solutions (2.10), restricting ourselves to analysing the stability in the linear approximation. In the process of constructing the periodic solutions (2.10) we have assumed that when $\mu = 0$, then the system (1.1) has no first order resonances except $\omega_m = N_0$. In considering the problem of stability we shall also assume that the system (1.1) has also no second order resonances when $\mu = 0$, i.e.

$$\sigma_i\omega_i + \sigma_j\omega_j \neq \pm N \quad (N = 1, 2, \ldots; \ i, j = 1, \ldots, n) \qquad (4.1)$$

and in the case $i = j$ we have $i \neq m$. This implies the absence of multiple resonances in the system, hence the stability of periodic solutions described by the formulas (2.10) will be determined by the part of the Hamiltonian function of perturbed motion depending on the variables with index m. We study the stability by introducing the variables q'' and p'' with help of the formulas $q'' = \sqrt{2R}\sin\psi$, $p'' = \sqrt{2R}\cos\psi$. The Hamiltonian function (2.6) now assumes the form

$$H'' = \sigma_m \varepsilon \frac{q''^2 + p''^2}{2} + l_{mm}\left(\frac{q''^2 + p''^2}{2}\right)^2 + \frac{\mu\delta}{2} p'' \tag{4.2}$$

and the equilibrium solution of the system (2.5) can be written in the form

$$q_0'' = 0, \quad p_0'' = \sqrt{2R_0}\cos\psi_0 = z$$

Carrying now in the Hamiltonian (4.2) the variables change according to the formulas $q'' = y$, $p'' = p_0'' + Y$, we obtain the part of the quadratic form of the Hamiltonian function of perturbed motion, which depends on the variables with index m, in the form

$$H_2'' = {}^1/_2(\sigma_m\varepsilon + l_{mm}p_0''^2)\,y^2 + {}^1/_2(\sigma_m\varepsilon + 3l_{mm}p_0''^2)\,Y^2$$

From this it follows that the necessary and sufficient condition for the stability of the periodic solutions (2.10) in the first approximation, in the absence in the system of resonances of the type (4.1), is that the following inequality holds:

$$(\sigma_m\varepsilon + l_{mm}z^2)\,(\sigma_m\varepsilon + 3l_{mm}z^2) > 0 \tag{4.3}$$

where z is given by (2.8). From the condition (4.3) it clearly follows that in the case of an exact resonance (i.e when $\varepsilon = 0$), a unique periodic motion described by the formulas (2.10) is always stable in the first approximation.

Let $\varepsilon \neq 0$. If $\sigma_m\varepsilon l_{mm} > 0$ (i.e. $D > 0$), then (4.3) implies that the unique 2π-periodic solution existing in this case is always stable. The case $\sigma_m\varepsilon l_{mm} < 0$ should be investigated by the simultaneous study of the condition (4.3) and equation (2.8). Let us give a brief description of the results of investigating the stability of periodic solutions described by the formulas (2.10). If $D > 0$, then the unique periodic solution which exists, is stable for any value of ε. When $D < 0$, three periodic solution (2.10) exist, with the amplitudes $|z_k|$ $(k = 1, 2, 3)$. We shall assume that $z_1 > z_2 > z_3$. Then for $\mu l_{mm} > 0$ the periodic solution (2.10) with amplitude $|z_1|$ is unstable, and solutions with the amplitudes $|z_2|$, $|z_3|$ are stable. If $\mu l_{mm} < 0$, then the periodic solutions with the amplitudes $|z_1|$, $|z_2|$ are stable and the solution with the amplitude $|z_3|$ is unstable. The results of investigating the stability of periodic solutions (2.10) are shown in the Fig.1 in the plane of parameters ε, μ $(\sigma_m = +1)$. The crosshatched regions have three periodic solutions. When ε and μ are in region $3a$, the periodic solution with the amplitude $|z_1|$ is unstable and those with the amplitudes $|z_2|$, $|z_3|$ are stable. In the regions 3b on the other hand, the periodic solutions with the amplitudes $|z_1|$, $|z_2|$ are stable, and that with the amplitude $|z_3|$ is unstable. On passing across the branch curves $D = 0$ from the crosshatched region of parameters ε, μ into the regions 1, the unstable and stable periodic solutions merge at the branching line and vanish in the regions 1.

5. In conclusion we shall describe, step by step, the procedure of constructing and investigating the stability of periodic solutions in the case of a resonance present in the forced oscillations.

1) determination of the problem's parameter values eliminated from the discussion, for which the initial system (1.1) has resonance relations of up to and including the fourth power, described by the formulas (1.3), (1.4) and (4.1);

2) normalization of the forms Γ_2, Γ_3, Γ_4 of the Hamiltonian (1.2) (determination of the coefficients a_{sm}, b_{sm} and l_{mm});

3) computation of f and g by means of the formulas (1.6);

4) determination of the quantities $\sin\theta_0$, $\cos\theta_0$, δ using the formulas (2.4);

5) derivation of the branching curve equation using the formula (2.9);

6) determination of the amplitude $|z|$ of periodic solutions (2.10) using the equation (2.8);

7) computation of the resonant periodic solutions from the formulas (2.10);

8) computation of 2π-periodic solutions in the initial variables ξ_i, η_i;

9) determination of the stability in the first approximation of periodic solutions obtained using Sect.4 and the Fig.1.

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